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Journal of Sound and Vibration 284 (2005) 985–1002
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Order reduction of nonlinear systems with time periodic coefficients using invariant manifolds

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Received 7 July 2003; received in revised form 19 July 2004; accepted 24 July 2004
Available online 16 December 2004

Abstract

The basic problem of order reduction of nonlinear systems with time periodic coefficients is considered. First, the equations of motion are transformed using the Lyapunov–Floquet transformation such that the linear parts of new set of equations are time invariant. At this stage, the linear order reduction technique can be applied in a straightforward manner. A nonlinear order reduction methodology is also suggested through a generalization of the invariant manifold technique via '*Time Periodic Center Manifold Theory*'. A '*reducibility condition*' is derived to provide conditions under which a nonlinear order reduction is possible. Unlike perturbation or averaging type approaches, the parametric excitation term is not assumed to be small. An example consisting of two parametrically excited coupled pendulums is given to show potential applications to real problems. Order reduction possibilities and results for various cases including 'parametric', 'internal', 'true internal' and 'combination' resonances are discussed.

1. Introduction

Many structural systems are modeled using the finite element technique. In this process, the dynamic response problem is reduced to a large set of differential equations. These equations may be linear or nonlinear where nonlinearities can arise from geometry or material behavior. An

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doi:10.1016/j.jsv.2004.07.027

important class of problems gives rise to a large number of equations with time varying coefficients. For the purpose of modal analysis, control and model testing, only a few dominating modes are important. Therefore, reduced order models that approximate the dynamics of original large-scale system, using linear as well as nonlinear reduction techniques, are needed.

Several linear approaches have been proposed to construct reduced order models for time invariant systems (of dimension r) that approximate the dynamics of the actual higher-order system of dimension n. The response of the reduced order model of dimension r ($\ll n$) is equivalent to the original large-scale system in some desired sense [1]. One of the techniques that developed primarily within the discipline of finite element analysis is called the '*Guyan reduction*' [2]. This is a linear static reduction procedure, which provides a reduced order model with coordinates that are a subset of the original coordinate system. Recently, '*Guyan reduction*' has been modified to account for inertia as well as stiffness properties. Other linear reduction methods, such as the *internal balancing technique*, are accomplished in state-space form and are more common to control applications. Other techniques related to control are described by Shokoohi et al. [3]. Guyan-like methods, while exact for linear systems, may also be applied to nonlinear systems, in which case the order reduction transformation is correct only for the linear terms. While such a linear strategy may yield acceptable results, the effects of the nonlinear terms are neglected in the order reduction process.

In order to construct a nonlinear order reduction technique for nonlinear systems, the concept of *nonlinear normal modes* (NNMs) may be utilized. As suggested by Shaw and Pierre [4], the nonlinear normal modes are defined as motions occurring on invariant manifolds, which are generally tangent to the corresponding eigenvectors of the linearized system at the equilibrium position and can be obtained analytically in a series form by various techniques [5]. The invariance property ensures that any motion starting exactly in a given modal manifold will remain in that manifold. One may perform a nonlinear modal analysis in order to obtain the system response in terms of some nonlinear modal coordinates. Model reduction using the nonlinear modal coordinates is advantageous because one may use fewer nonlinear normal modes than linear ones to perform equally accurate modal analysis of nonlinear systems. The order reduction methods may be carried out in state-space [6] or in second-order (structural) form [7]. In a recent paper, Burton and Rhee [8] suggested a nonlinear normal mode based order reduction method in structural form and compared it with linear reduction procedure for time invariant system.

In most cases, the original large-scale system of equations contains constant coefficients and methods discussed above can be used for order reduction. However, a very important class of problems, such as the dynamics of rotating systems, like helicopter blades, asymmetric rotor-bearing systems, and structures subjected to periodic loadings, etc., gives rise to equations with time periodic coefficients. Recently, order reduction techniques for linear time periodic systems have been reported using approximate multipoint Krylov techniques and time varying Padé approximation [9] from a control system perspective.

In the area of structural dynamics of time varying linear and nonlinear systems, very little work has been reported. In an isolated study [10], modal analysis and order reduction was performed on a rotor dynamic problem without taking into account the contribution of periodic terms. In general, such reduced models cannot portray the correct dynamics of the original periodic systems, to say the least. For a more meaningful approach, the contribution of periodic terms must be included in the transformation matrix. This may be accomplished by the use of Lyapunov–Floquet (L–F) transformation that converts the time varying linear system matrix into an equivalent time invariant form. The details of computation and application of L–F transformation can be found in Refs. [11,12]. Deshmukh et al. [13] recently applied this concept to develop an order reduction technique and control strategy for large-scale time periodic systems. Control laws based on modal analysis and aggregations were developed in the original coordinates by applying well-known control strategies to the reduced system in the transformed domain. A similar approach to nonlinear time periodic system may not yield acceptable results.

Very recently, Sinha et al. [14] have indicated that a nonlinear order reduction is possible for time periodic systems through an application of L–F transformation and invariant manifold concept. As in the case of '*Time Periodic Center Manifold Theory*', where 'stable' states are expressed as periodically modulated functions of 'critical' states, we propose to express the 'slave' (non-dominant) states as nonlinear functions of the 'master' (dominant) states. This is a generalization of the idea suggested by Shaw and Pierre [4] for the autonomous systems. The approach here implies the existence of invariant manifolds for the time-periodic dynamical systems.

The paper is divided into five sections. In Section 2, an introduction of Floquet theory and L–F transformation is presented while Section 3 outlines the general order reduction procedure. An example is presented in Section 4, which demonstrates the application and comparison with linear order reduction methodology. Discussion and conclusions are presented in Section 5.

2. Floquet theory and L–F transformation

Consider the linear periodic system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \qquad \mathbf{A}(t+T) = \mathbf{A}(t), \tag{1}$$

where $\mathbf{x}(t)$ is an n vector and $\mathbf{A}(t)$ is an $n \times n$ periodic matrix with the principal period T.

The state transition matrix (STM) $\mathbf{\Phi}(t)$ of Eq. (1) can be factored as [15]

$$\mathbf{\Phi}(t) = \mathbf{Q}(t)\mathrm{e}^{\mathbf{R}t}, \quad \mathbf{Q}(t) = \mathbf{Q}(t+2T), \quad \mathbf{Q}(0) = \mathbf{I}, \tag{2}$$

where the matrix $\mathbf{Q}(t)$ is real and periodic with period $2T$, $\mathbf{R}$ is an $n \times n$ real time invariant matrix and $\mathbf{I}$ is the identity matrix. Matrix $\mathbf{Q}(t)$ is known as the L–F transformation matrix [11].

The transformation $\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{z}(t)$ produces a real time invariant representation given by

$$\dot{\mathbf{z}}(t) = \mathbf{R}\mathbf{z}(t). \tag{3}$$

Thus, the nonlinear equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(\mathbf{x}, t) \tag{4}$$

may be converted to an equivalent form given by

$$\dot{\mathbf{z}} = \mathbf{R}\mathbf{z} + \mathbf{Q}^{-1}(t)\mathbf{f}(\mathbf{z}, t). \tag{5}$$

It is to be noted that the matrix $\mathbf{R}$, shown in Eq. (5), is time invariant.

3. Model order reduction techniques

3.1. The linear method

Now consider a multidimensional system described by a set of second-order nonlinear differential equations with time periodic coefficients as

$$\mathbf{M}(t)\ddot{\mathbf{y}} + \mathbf{C}(t)\dot{\mathbf{y}} + \mathbf{K}(t)\mathbf{y} + \mathbf{F}(\mathbf{y}, \dot{\mathbf{y}}, t) = 0, \tag{6}$$

where $\mathbf{y}$ is an m vector and $\mathbf{M}(t), \mathbf{C}(t), \mathbf{K}(t)$ are time periodic $m \times m$ $(m = n/2)$ matrices. $\mathbf{C}(t) = \mathbf{D}(t) + \mathbf{G}(t)$, where $\mathbf{D}(t)$ and $\mathbf{G}(t)$ are damping (symmetric) and gyroscopic (asymmetric) matrices, respectively, and $\mathbf{F}(\mathbf{y}, \dot{\mathbf{y}}, t)$ is a nonlinear vector function such that $\mathbf{F}(\mathbf{0}, \mathbf{0}, t) = \mathbf{0}$. The system described by Eq. (6) can be expressed as a set of n $(= 2m)$ first-order equations given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(\mathbf{x}, t), \tag{7}$$

where

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}(t)\mathbf{K}(t) & -\mathbf{M}^{-1}(t)\mathbf{C}(t) \end{bmatrix}.$$

$\mathbf{I}$ and $\mathbf{0}$ are $m \times m$ identity and null matrices, respectively, and $\mathbf{x}$ is an n vector of the states and $\mathbf{f}(\mathbf{x}, t)$ is a nonlinear time periodic n vector such that $\mathbf{f}(0, t) = 0$.

Applying the L–F transformation $\mathbf{x}(t) = \mathbf{Q}(t)\bar{\mathbf{y}}(t)$ produces

$$\dot{\bar{\mathbf{y}}}(t) = \mathbf{R}\bar{\mathbf{y}}(t) + \mathbf{Q}^{-1}(t)\mathbf{f}(\bar{\mathbf{y}}, t), \tag{8}$$

where $\mathbf{Q}(t)$ is the L–F transformation matrix of dimension $n \times n$.

After the modal transformation, $\bar{\mathbf{y}}(t) = \mathbf{M}\mathbf{z}(t)$, we have

$$\dot{\mathbf{z}}(t) = \mathbf{J}\mathbf{z}(t) + \mathbf{M}^{-1}\mathbf{Q}^{-1}(t)\mathbf{f}(\mathbf{z}, t) \equiv \mathbf{J}\mathbf{z}(t) + \mathbf{w}(\mathbf{z}, t), \tag{9}$$

where $\mathbf{J}$ is the Jordan form of $\mathbf{R}$ and $\mathbf{w}(\mathbf{z}, t)$ represents an appropriately defined nonlinear time varying vector consisting of monomials of z_j. The objective of order reduction is to replace the nonlinear time periodic system given by Eq. (9) by an equivalent system given by

$$\dot{\mathbf{z}}_r(t) = \mathbf{J}_r\mathbf{z}_r(t) + \bar{\mathbf{w}}_r(\mathbf{z}_r, t), \tag{10}$$

where $\mathbf{z}_r$ is an r $(r \ll n)$ vector of the dominant states to be retained, $\mathbf{J}_r$ is an $r \times r$ Jordan block corresponding to the dominant states and $\bar{\mathbf{w}}_r(\mathbf{z}_r, t)$ is a nonlinear vector function of appropriate dimensions consisting only of the dominant states.

At this stage Eq. (9) may be partitioned as

$$\begin{Bmatrix} \dot{\mathbf{z}}_r \\ \dot{\mathbf{z}}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{J}_r & 0 \\ 0 & \mathbf{J}_s \end{bmatrix} \begin{Bmatrix} \mathbf{z}_r \\ \mathbf{z}_s \end{Bmatrix} + \begin{Bmatrix} \mathbf{w}_r(\mathbf{z}_r, \mathbf{z}_s, t) \\ \mathbf{w}_s(\mathbf{z}_r, \mathbf{z}_s, t) \end{Bmatrix} \tag{11a,b}$$

where $\mathbf{z}_s$ is an $(n - r)$ vector of non-dominant states, $\mathbf{J}_s$ is the Jordan block of dimension $(n - r) \times (n - r)$ corresponding to the non-dominant states and $\mathbf{w}_r(\mathbf{z}_r, \mathbf{z}_s, t)$ and $\mathbf{w}_s(\mathbf{z}_r, \mathbf{z}_s, t)$ consist of the monomials of $\mathbf{z}$ (of order i) with periodic coefficients. In the linear approach, the contribution of the non-dominant states is considered insignificant and hence neglected. Thus, by neglecting Eq. (11b), the entire system dynamics is approximated by

$$\dot{\mathbf{z}}_r(t) = \mathbf{J}_r\mathbf{z}_r(t) + \mathbf{w}_r(\mathbf{z}_r, \mathbf{z}_s, t). \tag{12}$$

Eq. (12) is further approximated by setting the non-dominant states to zero ($\mathbf{z}_s = 0$), which yields

$$\dot{\mathbf{z}}_r(t) = \mathbf{J}_r\mathbf{z}_r(t) + \mathbf{w}_r(\mathbf{z}_r, 0, t). \tag{13}$$

Eq. (13) is the reduced order model of the actual large-scale system described by Eq. (9). Eq. (13) can be integrated numerically and using the transformation $\mathbf{x}(t) = \mathbf{Q}(t)\mathbf{MTz}_r(t)$, where $\mathbf{T} = [\mathbf{I}_{r\times r} \quad \mathbf{0}_{r\times(n-r)}]^{\mathrm{T}}$, all the states in $\mathbf{x}$ can be recovered.

The linear technique is simple and easy to implement. However, it does not give a clear insight if the dynamics is complex and involves internal and/or parametric resonance. The approximation error is larger and sometimes the results could be misleading.

3.2. Order reduction using time periodic invariant manifold

This methodology is based on the concept of *invariant manifold* for time-periodic systems. It implies that there exists a nonlinear time periodic relationship between dominant and non-dominant states, and therefore it is possible to reduce a large-scale system to a smaller system represented only by the dominant (master) states.

Consider the nonlinear time periodic system given by Eq. (11a,b). In this approach, a nonlinear relationship is assumed between the non-dominant ($\mathbf{z}_s$) and dominant ($\mathbf{z}_r$) states as

$$\mathbf{z}_s = \sum_i \mathbf{h}_i(\mathbf{z}_r, t) \equiv \mathbf{H}(\mathbf{z}_r, t), \tag{14}$$

where

$$\mathbf{h}_i = \sum_{\bar{\mathbf{m}}} \bar{\mathbf{h}}_i(t)\mathbf{z}_1^{m_1}\ldots\mathbf{z}_r^{m_r}, \quad \bar{\mathbf{m}} = (m_1,\ldots,m_r)^{\mathrm{T}}, \;\; m_1+\cdots+m_r = i, \;\; i = 2,3,\ldots,k. \tag{15}$$

Here $\bar{\mathbf{h}}_i(t)$ are the unknown periodic vector coefficients with period $2T$. Substitution of Eq. (14) into Eq. (11a,b) yields

$$\frac{\partial\mathbf{H}}{\partial t} + \frac{\partial\mathbf{H}}{\partial\mathbf{z}_r}(\mathbf{J}_r\mathbf{z}_r + \mathbf{w}_r) = \mathbf{J}_s\mathbf{H} + \mathbf{w}_s. \tag{16}$$

At this point, it should be observed that, if we consider the ith order $\mathbf{H}$ in the above equation, then $\mathbf{w}_s$ has to be approximated to the ith order as well, and we represent this by $\mathbf{w}_{si}$. However, all terms in $(\partial\mathbf{H}/\partial\mathbf{z}_r)\mathbf{w}_r$ are of the order $i+1$ or higher. Therefore, neglecting this product, we obtain

$$\frac{\partial\mathbf{H}}{\partial t} + \frac{\partial\mathbf{H}}{\partial\mathbf{z}_r}\mathbf{J}_r\mathbf{z}_r - \mathbf{J}_s\mathbf{H} = \mathbf{w}_{si}. \tag{17}$$

In order to solve this partial differential equation approximately, we expand the known and the unknown periodic coefficient functions ($\bar{\mathbf{h}}_i(t)$) into finite Fourier series as

$$\mathbf{h}_i(\mathbf{z}_r, t) = \sum_{j=1}^{s}\sum_{\bar{\mathbf{m}}}\sum_{v=-\infty}^{\infty} h_{j\bar{m}v}\,\mathrm{e}^{\bar{\mathrm{i}}v\pi t/T}|\mathbf{z}_r|^{\underset{\sim}{\mathbf{m}}}e_j, \tag{18}$$

$$\mathbf{w}_{si}(\mathbf{z}_r, t) = \sum_{j=1}^{s} \sum_{\bar{\mathbf{m}}} \sum_{v=-\infty}^{\infty} a_{j\bar{m}v} \mathrm{e}^{\bar{\mathrm{i}}v\pi t/T} |\mathbf{z}_r|^{\underset{\sim}{\mathbf{m}}} e_j, \tag{19}$$

where $|\mathbf{z}_r|^{\underset{\sim}{\mathbf{m}}} = z_1^{m_1} z_2^{m_2} \ldots z_r^{m_r}, \bar{\mathrm{i}} = \sqrt{-1}, m_1 + \cdots + m_r = i;\; i = 2, 3, \ldots, k.$ $a_{j\bar{m}v}$ are the known Fourier coefficients of the periodic functions, whereas $h_{j\bar{m}v}$ are the unknown Fourier coefficients of the manifold relation and e_j is the jth member of the natural basis. A term-by-term comparison of the Fourier coefficients yields

$$h_{jmv} = \frac{a_{jmv}}{(\bar{\mathrm{i}}v\pi/T) + \sum_{l=1}^{r} (m_l \lambda_l) - \bar{\lambda}_p}, \tag{20}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the eigenvalues of the Jordan matrix $\mathbf{J}_r$ and $\bar{\lambda}_p$; $p = 1, 2, \ldots, s$ are the eigenvalues of $\mathbf{J}_s$. Therefore, the '*reducibility condition*' is expressed as

$$\frac{\bar{\mathrm{i}}v\pi}{T} + \sum_{l=1}^{r} (m_l \lambda_l) - \bar{\lambda}_p \neq 0 \quad \forall v = 0, \pm 1, \pm 2 \ldots;\; p = 1, 2, \ldots, s. \tag{21}$$

It is obvious that if the '*reducibility condition*' is satisfied then the vector $\mathbf{H}(\mathbf{z}_r, t)$ can be obtained and the 'slave' states can be expressed in terms of the 'master' states. However, when this condition is not satisfied, such a reduction is not possible and 'slave' coordinates cannot be expressed as functions of 'master' coordinates.

For $v = 0$, a resonance occurs when some linear combination of λ_l (frequencies of the 'master' states) and $\bar{\lambda}_p$ (frequencies of the 'slave' states) add up to zero. This may be referred to as the '*true internal resonance*'. If the second-order system given by Eq. (6) is autonomous (i.e., $\mathbf{M}$ ($= \mathbf{M}^{\mathrm{T}}$), $\mathbf{C}$, $\mathbf{K}$ ($= \mathbf{K}^{\mathrm{T}}$) and $\mathbf{f}$ do not explicitly depend on time t) and $\mathbf{D}$ is identically zero, then $\lambda_n = \pm \mathrm{i}\omega_n$ (where ω_n are the natural frequencies corresponding to the original coordinates in $\mathbf{y}$), and Eq. (21) provides conditions for the conventional '*internal resonance*' widely discussed in the literature. Shaw et al. [6] derived such conditions for the special case of quadratic and cubic nonlinearities using **Mathematica**$^{\mathbf{TM}}$. It can be easily verified that their result is a sub set of the results contained in Eq. (21). For $v \neq 0$, the denominator in Eq. (20) goes to zero when the parametric excitation frequency ω ($= 2\pi/T$) is a linear combination of λ_l and $\bar{\lambda}_p$. This situation is referred to as the '*true combination resonance*' case. Further, let us look at a special form of Eq. (6) such that $\mathbf{M}(t) = [\mathbf{M}_0 + \varepsilon \mathbf{M}_1(t)]$, $\mathbf{C}(t) \equiv \mathbf{G}(t) = [\mathbf{G}_0 + \varepsilon \mathbf{G}_1(t)]$, $\mathbf{K}(t) = [\mathbf{K}_0 + \mathbf{K}_1(t)]$ and $\mathbf{F}(\mathbf{y}, \dot{\mathbf{y}}, t) = \varepsilon \mathbf{F}(\mathbf{y}, \dot{\mathbf{y}}, t)$. It is interesting to observe that if the coefficients (ε) of parametric excitation and nonlinear terms are small, then the eigenvalues $\lambda_n \to \pm \mathrm{i}\omega_n$, and from Eq. (20) we recover the condition for conventional '*combination resonance*' in parametrically excited nonlinear systems with a small parameter [16]. The concept of '*parametric resonance*' comes from the stability analysis of linear systems with time periodic coefficients. In this case at least one pair of Floquet multipliers is repeated. For a single degree of freedom system, the repeated multipliers have to be real and repeated as -1 or $+1$. This implies that at least one pair of eigenvalues of the $\mathbf{J}_r$ matrix has a zero value. For multiple degrees of freedom systems, two pairs of Floquet exponents have to be purely imaginary and repeated. For systems with small parametric excitation terms (as discussed above), it takes the form of sum and difference type of resonance conditions. The details may be found in Refs. [17,18]. If $\mathbf{J}_r$ matrix contains a pair of zeros (critical eigenvalues), then the order reduction process necessarily boils down to finding a time-periodic center manifold relation between the 'slave' and the 'master' states. Since Floquet multipliers are either -1 or $+1$, the system undergoes a 'flip' or a symmetry breaking (or transcritical) bifurcation, respectively. The reduced order model does not contain any linear terms; it is strictly a nonlinear model. Once $\mathbf{H}(\mathbf{w}_r, t)$ has been determined, we obtain the equation for the 'master' states $\mathbf{z}_r$ as

$$\dot{\mathbf{z}}_r = \mathbf{J}_r\mathbf{z}_r + \bar{\mathbf{w}}_r(\mathbf{z}_r, \mathbf{H}(\mathbf{z}_r, t), t). \tag{22}$$

This is the reduced order system in the $\mathbf{z}$ domain. Now we can make use of the L–F transformation matrix $\mathbf{Q}(t)$ and the modal matrix $\mathbf{M}$ to map the results back to the original $(\mathbf{x})$ domain. Since this transformation is a Lyapunov transformation, all stability properties are preserved and the existence of invariant manifold is guaranteed.

4. Applications

The techniques proposed in Section 3 can be effectively applied to reduce the order of practical engineering structures modeled by nonlinear differential equations with time periodic coefficients. To demonstrate possible practical applications, we reduce the order of a system consisting of two inverted coupled pendulums moving in the horizontal plane with time-dependent load acting on each pendulum. Each pendulum is supported at the base by a torsional spring. The loading consists of time periodic concentrated axial loads. The structural diagram of the system considered is shown in Fig. 1. The equations of motion can be shown to be

$$ml^2\ddot{\theta}_1 + k_{t1}\theta_1 + k\frac{l^2}{4}q_1(\theta_1, \theta_2) - P_1(t)\sin\theta_1 = 0, \tag{23a}$$

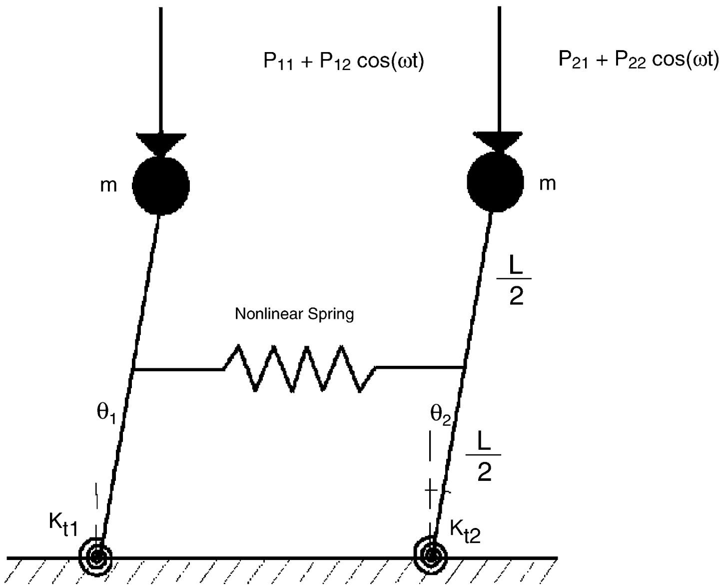

Fig. 1. Coupled pendulums.

$$ml^2\ddot{\theta}_2 + k_{t2}\theta_2 + k\frac{l^2}{4}q_2(\theta_1,\theta_2) - P_2(t)\sin\theta_2 = 0, \tag{23b}$$

where $q_1(\theta_1,\theta_2)$ and $q_2(\theta_1,\theta_2)$ are nonlinear functions of $(\theta_1-\theta_2), P_1(t) = P_{11} + P_{12}\cos(\omega t)$ and $P_2(t) = P_{21} + P_{22}\cos(\omega t)$. m, l, k_t and k denote mass, length, torsional stiffness and coupling stiffness, respectively. The local dynamics can be obtained by expanding these global equations of motion about the fixed point $(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2) = (0,0,0,0)$. The linearized equations can be written as

$$\ddot{\theta}_1 + \frac{k_{t1}}{ml^2}\theta_1 + \frac{k}{4m}(\theta_1-\theta_2) - \left[\frac{P_{11}l}{ml^2} + \frac{P_{12}l}{ml^2}\cos(\omega t)\right](\theta_1) = 0,$$

$$\ddot{\theta}_2 + \frac{k_{t1}}{ml^2}\theta_2 + \frac{k}{4m}(\theta_2-\theta_1) - \left[\frac{P_{21}l}{ml^2} + \frac{P_{22}l}{ml^2}\cos(\omega t)\right](\theta_2) = 0. \tag{24}$$

On the other hand, if terms up to the cubic order are retained then these equations may be approximated as

$$\begin{aligned}\ddot{\theta}_1 &+ \frac{k_{t1}}{ml^2}\theta_1 + \frac{k}{4m}[c_1(\theta_1-\theta_2) + c_2(\theta_1-\theta_2)^2 + c_3(\theta_1-\theta_2)^3]\\ &- \left[\frac{P_{11}l}{ml^2} + \frac{P_{12}l}{ml^2}\cos(\omega t)\right]\left(\theta_1 - \frac{\theta_1^3}{6}\right) = 0,\end{aligned}$$

$$\begin{aligned}\ddot{\theta}_2 &+ \frac{k_{t2}}{ml^2}\theta_2 + \frac{k}{4m}[c_1(\theta_2-\theta_1) + c_2(\theta_2-\theta_1)^2 + c_3(\theta_2-\theta_1)^3]\\ &- \left[\frac{P_{21}l}{ml^2} + \frac{P_{22}l}{ml^2}\cos(\omega t)\right]\left(\theta_2 - \frac{\theta_2^3}{6}\right) = 0,\end{aligned} \tag{25}$$

where c_1, c_2, c_3 are coupling parameters. Setting P_{11} and P_{21} equal to zero, Eq. (25) can be written as

$$\ddot{\theta}_1 + (\omega_{n_1}^2 - \varepsilon p_1\cos(\omega t))\theta_1 + \varepsilon p_1\cos(\omega t)\frac{\theta_1^3}{6} - b\theta_2 - c(\theta_1-\theta_2)^2 - d(\theta_1-\theta_2)^3 = 0, \tag{26a}$$

$$\ddot{\theta}_2 + (\omega_{n_2}^2 - \varepsilon p_2\cos(\omega t))\theta_2 + \varepsilon p_2\cos(\omega t)\frac{\theta_2^3}{6} - b\theta_1 + c(\theta_1-\theta_2)^2 + d(\theta_1-\theta_2)^3 = 0, \tag{26b}$$

where

$$\omega_{n_1}^2 = \left[\frac{k_{t1}}{ml^2} + \frac{k}{4m}c_1\right],\quad \omega_{n_2}^2 = \left[\frac{k_{t2}}{ml^2} + \frac{k}{4m}c_1\right],\quad \varepsilon p_1 = \frac{P_{12}l}{ml^2},\quad \varepsilon p_2 = \frac{P_{22}l}{ml^2},\quad \omega = 2\pi,$$

$$b = \frac{k}{4m}c_1,\quad c = \frac{k}{4m}c_2,\quad d = \frac{k}{4m}c_3.$$

Eq. (26) can be written in the state space form as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(\omega_{n_1}^2 - \varepsilon p_1 \cos(\omega t)) & b & 0 & 0 \\ b & -(\omega_{n_2}^2 - \varepsilon p_2 \cos(\omega t)) & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -\varepsilon p_1 \cos(\omega t) \frac{x_1^3}{6} + c(x_1 - x_2)^2 + d(x_1 - x_2)^3 \\ -\varepsilon p_2 \cos(\omega t) \frac{x_2^3}{6} - c(x_1 - x_2)^2 - d(x_1 - x_2)^3 \end{Bmatrix}, \tag{27}$$

where

$$\{x_1 \quad x_2 \quad x_3 \quad x_4\}^{\mathrm{T}} = \{\theta_1 \quad \theta_2 \quad \dot{\theta}_1 \quad \dot{\theta}_2\}^{\mathrm{T}}. \tag{28}$$

Applying the L–F transformation, Eq. (27) may be written as

$$\dot{\bar{\mathbf{y}}} = \mathbf{R}\bar{\mathbf{y}} + \mathbf{Q}^{-1}(t)\mathbf{f}(\bar{\mathbf{y}}, t), \tag{29}$$

where $\bar{\mathbf{y}} = \{\bar{y}_1\ \bar{y}_2\ \bar{y}_3\ \bar{y}_4\}^{\mathrm{T}}$ and $\mathbf{R}$ is a 4×4 time invariant matrix. By choosing a suitable set of values of coupling parameters (b, c, d), the system can be made to undergo 'parametric resonance', (3:1 and 2:1), 'internal resonance', 'true internal resonance' or 'true combination resonance'. These resonances are discussed in the context of this example.

4.1. Case 1: no resonance of any kind

The system parameters are chosen such that the system does not exhibit 'internal resonance', 'true internal resonance' or 'true combination resonance'. These parameters are given in Table 1. For this set of parameters, the Floquet multipliers are given as $(-0.51 \pm 0.85\mathrm{i}, -0.88 \pm 0.45\mathrm{i})$. The L–F transformation matrix $\mathbf{Q}(t)$ is computed from the linear part of Eq. (27) by the method suggested by Sinha et al. [11]. The $\mathbf{R}$ matrix corresponding to Eq. (29) is found to be

$$\mathbf{R} = \begin{bmatrix} 0.001 & 0 & -0.307 & -0.053 \\ 0 & 0 & -0.053 & -0.093 \\ 3.311 & 0.127 & 0 & 0 \\ 0.127 & 2.804 & 0 & 0 \end{bmatrix}. \tag{30}$$

Table 1
Parameter set for the case when no parametric/internal resonance exists

Parameter	$\omega_{n_1}^2$	$\omega_{n_2}^2$	εp_1	εp_2	ω	b	c	d
Value	3.5	5.5	5	5	2π	0.5	0	1.5

The Floquet exponents (eigenvalues of **R** matrix) are ($\pm 0.47i, \pm 1.03i$). It can be easily verified that the ratio of the angles of Floquet multipliers (or the ratio of Floquet exponents) for this set is 2.19, which implies they are not in 'true internal resonance' and the '*reducibility condition*' given by Eq. (21) is satisfied.

Using the modal transformation

$$\mathbf{y} = \mathbf{M}\mathbf{z}, \tag{31}$$

where **M** is the 4×4 modal matrix of **R** obtained by solving the eigenvalue problem and $\mathbf{z} = \{z_1\ z_2\ z_3\ z_4\}^{\mathrm{T}}$. Eq. (29) can be written explicitly as

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{Bmatrix} = \begin{bmatrix} 0 & -0.47 & 0 & 0 \\ 0.47 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.03 \\ 0 & 0 & 1.03 & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} + \begin{Bmatrix} w_1(\mathbf{z}, t) \\ w_2(\mathbf{z}, t) \\ w_3(\mathbf{z}, t) \\ w_4(\mathbf{z}, t) \end{Bmatrix}, \tag{32}$$

where $w_i(\mathbf{z}, t)$ are nonlinear functions with periodic coefficients comprising of all the states. The long expressions for $w_i(\mathbf{z}, t)$ are obtained using **Mathematica**$^{\mathbf{TM}}$ and omitted here for brevity. At this stage, we reduce the order of the system given by Eq. (32) using methods described in Section 3.

4.1.1. *Order reduction using linear method*

Eq. (32) comprises of 4 states $\{z_1\ z_2\ z_3\ z_4\}^{\mathrm{T}}$, out of those 4 states. We choose $\mathbf{z}_r = \{z_1\ z_2\}^{\mathrm{T}}$ as the dominant states; they correspond to the lowest eigenvalues of the system. By neglecting the contribution from the non-dominant states $\mathbf{z}_s = \{z_3\ z_4\}^{\mathrm{T}}$, the system dynamics can be expressed as

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} 0 & -0.47 \\ 0.47 & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{Bmatrix} w_1(z_1, z_2, z_3, z_4, t) \\ w_2(z_1, z_2, z_3, z_4, t) \end{Bmatrix}. \tag{33}$$

This system is further approximated by setting z_3, z_4 to zero to obtain

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} 0 & -0.47 \\ 0.47 & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{Bmatrix} w_1(z_1, z_2, 0, 0, t) \\ w_2(z_1, z_2, 0, 0, t) \end{Bmatrix}, \tag{34}$$

Eq. (34) is the reduced order model of the system described by Eq. (32), obtained using the linear method. This reduced order system is integrated numerically with some typical initial conditions and all the states in $\mathbf{x} = \{\theta_1\ \theta_2\ \dot{\theta}_1\ \dot{\theta}_2\}^{\mathrm{T}}$ are obtained using L–F and the modal transformations. In Fig. 2(a), the time trace of $\dot{\theta}_1$ obtained using the above procedure is compared with the time trace of $\dot{\theta}_1$ obtained by integrating the original Eq. (27). It can be seen that these time traces match well but not exactly. To portray the results from long time simulations, we plot the Poincaré maps of the original large-scale system and the reduced order system sampled at the frequency of parametric excitation. In Fig. 3(a), the Poincaré map of large-scale system is shown, and Fig. 3(b) shows the Poincaré map of reduced system using the linear method. The Poincaré map of the large-scale system shows a 'band' implying the existence of quasiperiodic motion. This 'band' is not so dominant in the map of the reduced

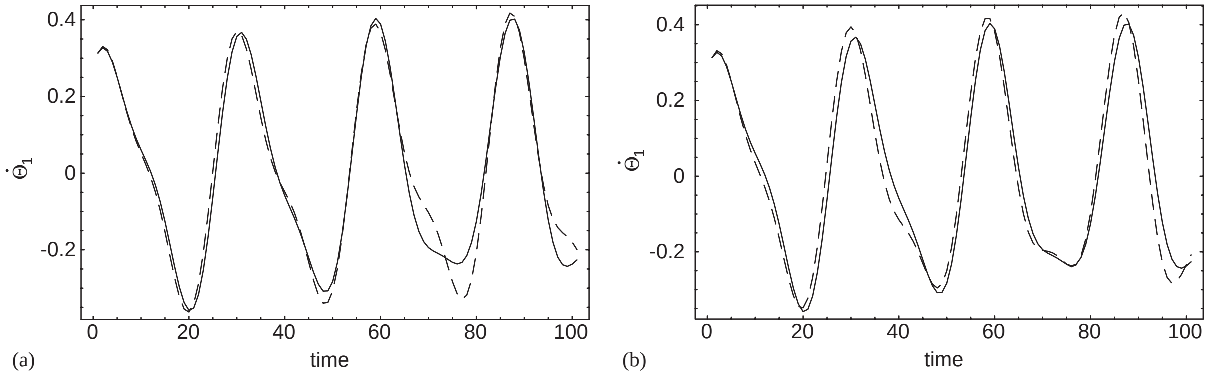

Fig. 2. Comparison of time traces of 'master velocities': (a) $\dot{\theta}_1$, ——; $\dot{\theta}_{1(\text{linear reduction})}$, – –, (b) $\dot{\theta}_1$, ——; $\dot{\theta}_{1(\text{invariant manifold reduction})}$ – – .

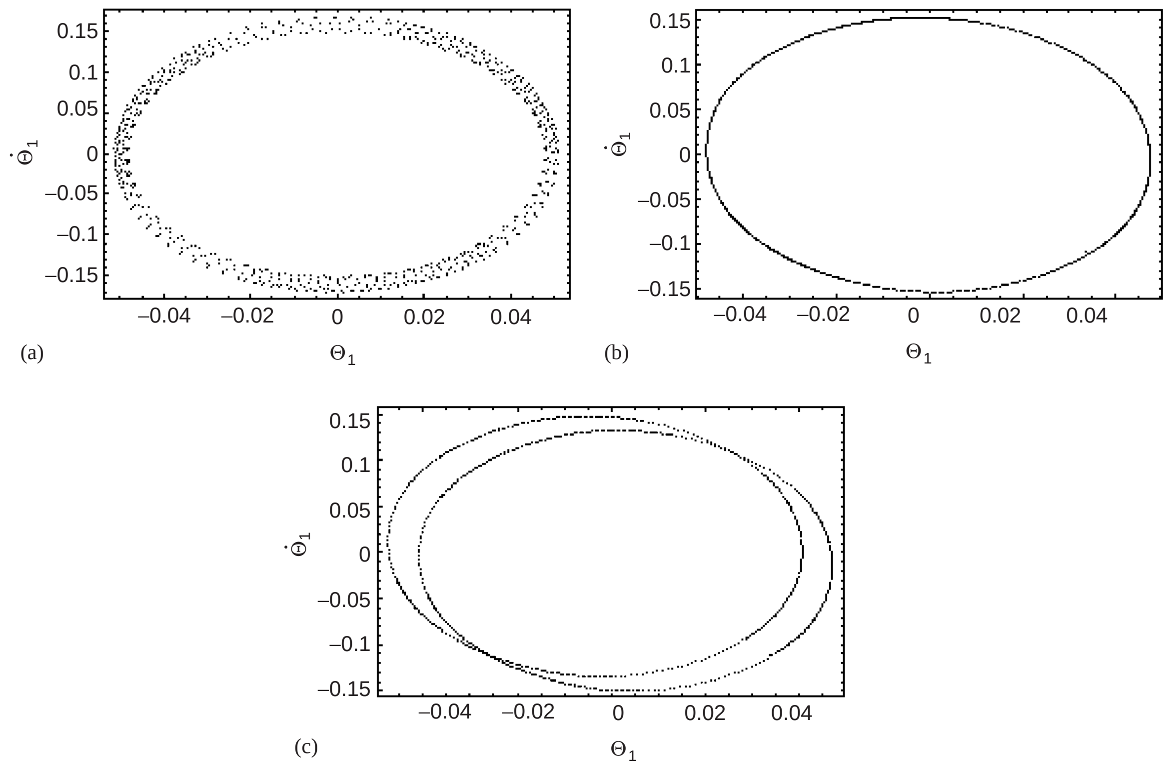

Fig. 3. Poincaré maps for the 'no resonance' case: (a) large-scale system, (b) reduced order system via linear method, (c) reduced order system via nonlinear method.

system using the linear method (the simulation time is same for both). These maps do not match exactly in magnitude as well. Thus, we conclude that the linear method does not yield an accurate reduced order model, at least for this particular case.

4.1.2. Order reduction using invariant manifold

As discussed in Section 3.2 we try to relate the non-dominant (slave) states to the dominant (master) states by a time periodic nonlinear transformation. If the system does not exhibit 'true internal resonance' (like the case under consideration) then the '*reducibility condition*' is satisfied and the system order can be reduced.

Once again, we start with Eq. (32) and choosing same states, $\mathbf{z}_r = \{z_1 \; z_2\}^{\mathrm{T}}$ as the dominant states, try to find a nonlinear time periodic relationship between the dominant and the non-dominant states as given by Eq. (14). The relationship between $\mathbf{z}_s$ and $\mathbf{z}_r$ is expressed as

$$\mathbf{z}_s = \sum_i \mathbf{h}_i(z_1, z_2, t) \equiv \mathbf{H}(z_1, z_2, t), \quad s = 3, 4, \tag{35}$$

where

$$\mathbf{h}_i = \sum_{\bar{\mathbf{m}}} \bar{\mathbf{h}}_i(t) z_1^{m_1} \ldots z_2^{m_2}, \;\; \bar{\mathbf{m}} = (m_1, m_2)^{\mathrm{T}}, \quad m_1 + m_2 = 3. \tag{36}$$

Here $\bar{\mathbf{h}}_i(t)$ are the unknown periodic vector coefficients with period $2T$. We substitute Eq. (35) into Eq. (32). After expanding $\bar{\mathbf{h}}_i(t)$ and $\mathbf{w}_s(z_r, t)$ $s = 3, 4$ in Fourier series, and neglecting the terms of higher order, we obtain the relationship between the dominant and the non-dominant states as

$$\begin{aligned} z_3 = {} & z_1 z_2^2(0.013 - 0.012\cos(2\pi t)) + z_1^3(0.048 - 0.044\cos(2\pi t) + 0.001\cos(4\pi t)) \\ & + z_2^3(0.041\sin(2\pi t)) + z_1^2 z_2(-0.022\sin(2\pi t) + 0.014\sin(4\pi t)) = \mathbf{H}_1(z_1, z_2, t), \end{aligned}$$

$$\begin{aligned} z_4 = {} & z_1 z_2^2(0.094 - 0.021\cos(2\pi t) - 0.055\cos(4\pi t)) + z_2^3(0.061) + z_1^3(-0.017\sin(2\pi t) \\ & - 0.063\sin(4\pi t)) + z_2 z_1^2(-0.023\sin(2\pi t) - 0.012\sin(4\pi t)) = \mathbf{H}_2(z_1, z_2, t). \end{aligned} \tag{37}$$

Eq. (37) is substituted into top half of Eq. (32) and the reduced order model is obtained as

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} 0 & -0.47 \\ 0.47 & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{Bmatrix} \bar{w}_1(z_1, z_2, t) \\ \bar{w}_2(z_1, z_2, t) \end{Bmatrix}. \tag{38}$$

This symbolic computation was carried out using **Mathematica**™ and the explicit long expressions are omitted here for brevity. As earlier, Eq. (38) is solved numerically and again by making use of L–F and modal transformations, all components of vector **x** can be constructed. The time trace of $\dot{\theta}_1$ (Fig. 2(b)) obtained by this method is compared with the time trace of $\dot{\theta}_1$ obtained by numerical integration of the original equation given by Eq. (27). As we can see, the two time traces match quite well. The Poincaré map of the original system (given by Fig. 3(a)) matches the Poincaré map of the reduced order system (given by Fig. 3(c)) quantitatively. It can also be observed that the 'band' in the Poincaré maps of the original system is approximated by 2 loops representing the most dominant frequencies in the quasiperiodic motion of the system dynamics.

The spectral plots corresponding to the original large-scale system and the reduced order systems by linear and invariant manifold methods are given in Figs. 4(a)–(c), respectively. It can be seen that all these spectral plots show two peaks corresponding to the dominant frequencies in the quasiperiodic motion exhibited by the system. However, the spectral plot of invariant manifold based reduced system (Fig. 4(c)) matches more accurately to the spectral plot of the

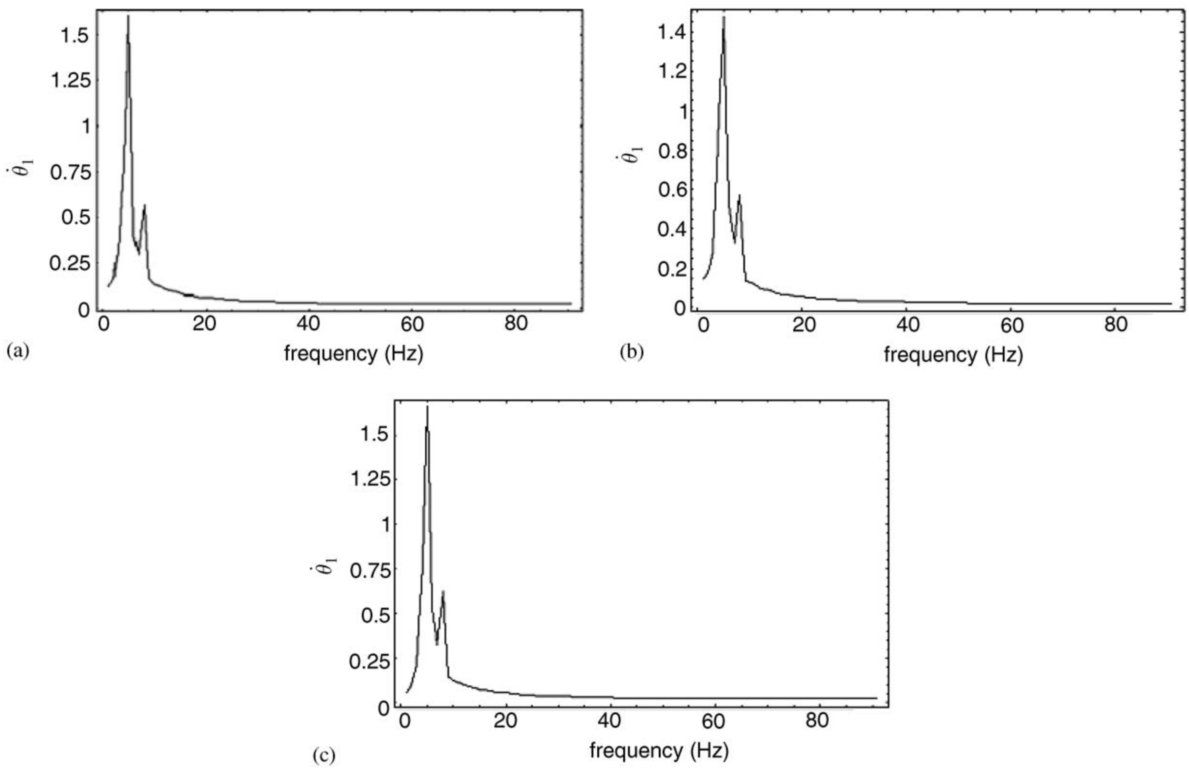

Fig. 4. Spectral plots for the 'no resonance' case: (a) large-scale system, (b) reduced order system via linear method, (c) reduced order system via nonlinear method.

Table 2
Parameter set for the case when parametric resonance exists

Parameter	$\omega_{n_1}^2$	$\omega_{n_2}^2$	εp_1	εp_2	ω	b	c	d
Value	355.3	20	5	5	2π	0	0	1

original system (Fig. 4(a)) than the reduced system obtained by the linear method (Fig. 4(b)) in magnitude. Therefore, we conclude that the invariant manifold order reduction technique yields better results when compared to the linear method.

4.2. Case 2: parametric resonance

As discussed in Section 3.2, if the parameter (ε) multiplying the periodic term of an undamped system is small and a pair of Floquet multipliers is real and repeated, then a 'parametric resonance' takes place. It is well known that the 'principal parametric resonance' corresponds to the case when the parametric excitation frequency ω and one of the natural frequencies ω_{ni} satisfy the ratio of 2:1. This case is known to be unstable. For numerical simulations, we choose the parameters given in Table 2. It can be seen as the linear coupling term is zero (i.e., $b = 0$) and the linear part of each equation in (26) resembles a Mathieu equation. By selecting $\omega_{n_1} \simeq 3\omega$, we observe that Eq. (26a) (equation in θ_1) indicates (1:3) resonance and the system is 'mildly' unstable. The Floquet multipliers are: $(0.99, 0.99, -0.27 + 0.96i, -0.27 - 0.96i)$. After the L–F and the modal transformation the system is given by

$$\begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.29 \\ 0 & 0 & 1.29 & 0 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{Bmatrix} + \begin{Bmatrix} w_1(\mathbf{z}, t) \\ w_2(\mathbf{z}, t) \\ w_3(\mathbf{z}, t) \\ w_4(\mathbf{z}, t) \end{Bmatrix}. \tag{39}$$

The zero entries (top left 2×2) in Eq. (39) correspond to the resonant mode and $w_i(\mathbf{z}, t)$ are nonlinear functions of states as defined earlier.

4.2.1. Order reduction using the linear method

This problem clearly shows the limitation of the linear method. Here the reduced order model obtained using linear method does not yield acceptable results due to the fact that the major contribution in the dynamics comes from the nonlinear terms and this method eliminates the significant contribution regardless of which variables are selected as dominant states. It occurs here due to the fact that this problem has no linear coupling and L–F and the modal transformations assume special forms.

4.2.2. Order reduction using invariant manifold

It can be shown that the '*reducibility condition*' for this problem is satisfied and order reduction is possible. Here, one may choose the resonant or the non-resonant modes as the states to be retained and eliminate the other modes. The resulting reduced order system exhibits very interesting and complex dynamics. If we choose $(\theta_1, \dot{\theta}_1)$ (the resonant mode) as the 'master' coordinates then we are essentially performing a center manifold reduction. The Poincaré maps for large-scale system and the reduced order system are shown in Fig. 5. They show a simple quasiperiodic motion due to the fact that in the $\mathbf{z}$ domain a limit cycle is born after this mode has gone through a symmetry breaking bifurcation. The limit cycle is shown in Fig. 7(a). Similar

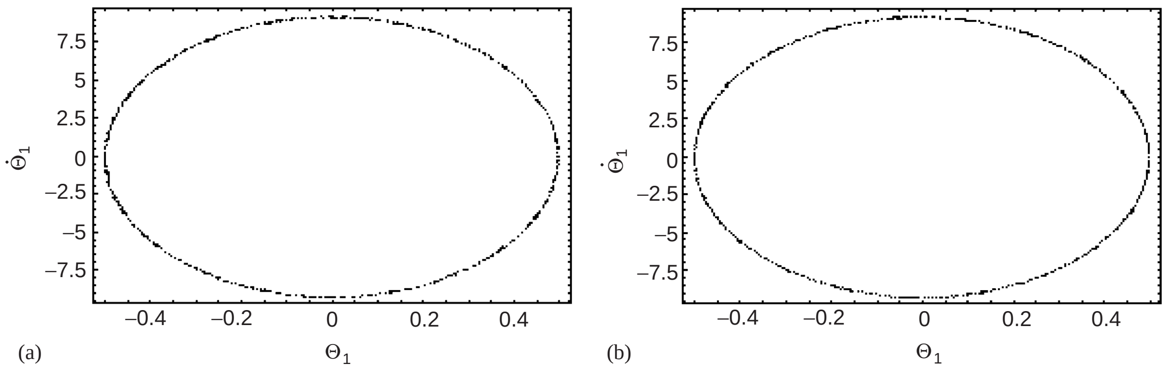

Fig. 5. Poincaré maps when 'resonant modes' are selected as 'master' coordinates: (a) large-scale system, (b) reduced order system via nonlinear method.

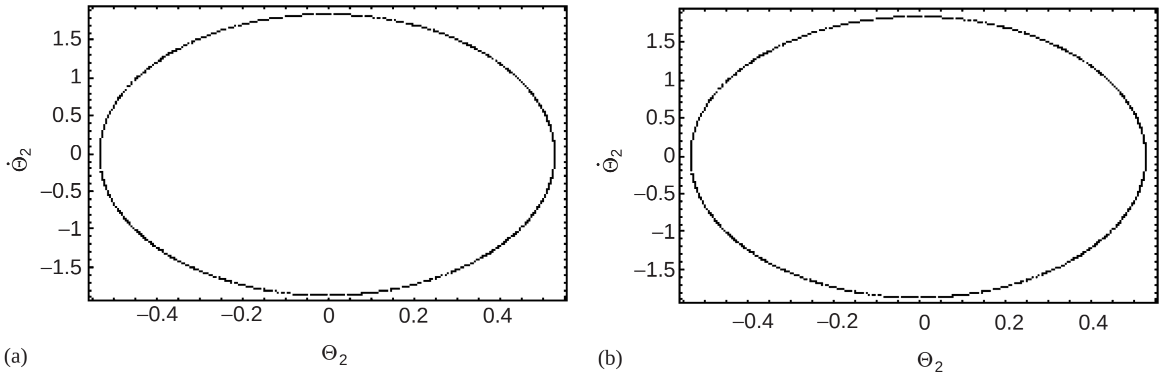

Fig. 6. Poincaré maps when 'non-resonant modes' selected as 'master' coordinates: (a) large-scale system, (b) reduced order system via nonlinear method.

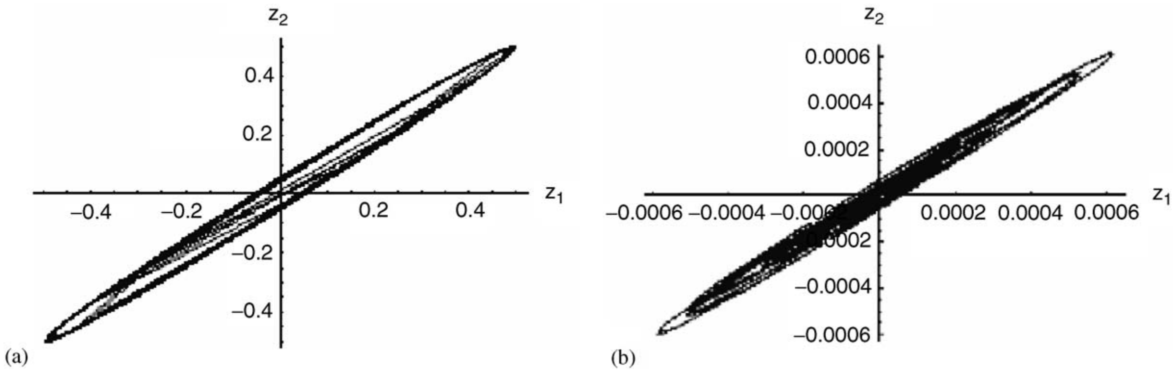

Fig. 7. Phase plane plot of master coordinates in '**z** domain': (a) 'resonant modes' are selected as the 'master' coordinates, (b) 'non-resonant modes' are selected as the 'master' coordinates.

results are obtained if we select $(\theta_2, \dot{\theta}_2)$ (non-resonant modes) as the 'master' coordinates and the results are shown in Fig. 6. The limit cycle in **z** domain corresponding to this mode is shown in Fig. 7(b).

For a general case when the quadratic coupling exists ($c \neq 0$), it can be shown that the '*reducibility condition*' is satisfied only when the resonant modes are selected as 'master coordinates' (see Ref. [19] for details).

4.3. Case 3: 'true internal resonance'

In the case of 'true internal resonance', there exists an irremovable coupling among the system states, the '*reducibility condition*' is not satisfied, and it is not possible to obtain a relationship between the dominant and the non-dominant states. For illustration, we select the system parameters as given in Table 3. For this set of parameters, the Floquet multipliers are (-0.96 ± 0.26i, 0.57 ± 0.81i) and the eigenvalues of **R** matrix are given by (± 0.95i, ± 2.86i). It is also to be

Table 3
Parameter values for the case when 'true internal resonance' exists

Parameter	$\omega_{n_1}^2$	$\omega_{n_2}^2$	εp_1	εp_2	ω	b	c	d
Value	4.5	4	0.1	4.5	2π	3.5	0	1

observed that the **H** vector has the form

$$H_j = a_{1j}(t)z_1^3 z_2^0 + a_{2j}(t)z_1^2 z_2^1 + a_{3j}(t)z_1^1 z_2^2 + a_{4j}(t)z_1^0 z_2^3, \quad j = 3, 4, \tag{40}$$

where $a_{ij}(t)$ are periodic functions of time.

In this case, the '*reducibility condition*' is not satisfied since for $v = 0$, Eq. (21) yields $3 \times 0.95\mathrm{i} - 2.86\mathrm{i} = 0$ (note that $m_1 = 3, m_2 = 0; \lambda_1, \lambda_2 = \pm 0.95\mathrm{i}$ and $\bar{\lambda}_1, \bar{\lambda}_2 = \pm 2.86\mathrm{i}$). Thus, the modes cannot be decoupled and order reduction is not possible. Of course, one can always perform a linear based order reduction that may yield acceptable results for very small initial conditions when the nonlinear effects are not so significant. However, in general, as the initial conditions are selected farther from the equilibrium position then none of the order reduction methodologies yield acceptable results.

5. Discussion and conclusions

In this paper, the order reduction problem of nonlinear systems with time periodic coefficients is considered. Here, the order reduction techniques are developed in the state space form of system equations. First, the equations of motion are transformed using the L–F transformation such that the linear parts of new set of equations are time invariant. At this stage, a linear order reduction technique is suggested to separate the 'master' (dominant) states from the 'slave' (non-dominant) states such that the dynamics of the 'master' states is a meaningful approximation of the full-scale system. A nonlinear order reduction methodology is also proposed through a generalization of the invariant manifold technique via *time periodic invariant manifold theory*. A nonlinear time periodic relationship between the 'slave' and the 'master' states is suggested and a '*reducibility condition*' is derived to determine whether a nonlinear order reduction is possible or not. The '*reducibility condition*' also provides more general definitions and interpretations of various types ('parametric', 'conventional internal', 'true internal', 'combination' and 'true combination') of resonances encountered in parametrically excited systems. Unlike perturbation or averaging type approaches, the parametric excitation term is not assumed small. An example consisting of two parametrically excited coupled pendulums is given to show possible applications to real engineering problems. Order reduction possibilities and results for various resonances are discussed. A Poincaré map is used as a measure to compare the accuracy of the reduced order models. The authors believe this is important because the Poincaré map truly depicts the long-term dynamical behavior of the system and is definitely a superior measure than comparing just the short time traces of the original and the reduced order systems. It is found that nonlinear order reduction techniques provide accurate approximations (compared to linear methods) of the full-scale system in the event when no resonances are occurring. Nonlinear order reduction is not possible when the system is subjected to either the '*true internal resonance*' or the '*true combination resonance*'. In these cases, the '*reducibility condition*' is not satisfied. One can always try the linear order reduction approach; however, the results may not be meaningful.

In conclusion, it can be stated that a rigorous technique for order reduction of general linear and nonlinear dynamical systems with time periodic coefficients is presented. The parametric terms appearing in the linear parts of system equations are not assumed to be small. A mathematical condition is also derived to determine whether (or not) a large-scale nonlinear parametrically excited system can be reduced to a lower order system. Several extensions, generalizations and applications (in the area of controls) of the methods described in this paper are in progress.

Acknowledgements

The financial support provided by the Air Force Office of Scientific Research (AFOSR) in collaboration with the University of Alaska Fairbanks under the Contract No. F49620- 01-1-0388 and National Science Foundation (Contract No. CMS 0114571) is gratefully acknowledged.

References

[1] M.S. Mahmoud, M.G. Singh, *Large Scale Systems Modelling*, Pergamon Press, Oxford, UK, 1981.
[2] R.J. Guyan, Reduction of stiffness and mass matrices, *AIAA Journal* 3 (1965) 380.
[3] S. Shokoohi, L. Silverman, P. Van Dooren, Linear time-variable systems: balancing and model reduction, *IEEE Transactions on Automatic Control* AC-28 (1983) 810–822.
[4] S.W. Shaw, C. Pierre, Nonlinear normal modes and invariant manifolds, *Journal of Sound and Vibration* 150 (1) (1991) 170–173.
[5] A.F. Vakakis, Non-linear normal modes (NNMs) and their applications in vibration theory: an overview, *Mechanical Systems and Signal Processing* 11 (1) (1997) 3–22.
[6] S.W. Shaw, C. Pierre, E. Pesheck, Modal analysis-based reduced-order models for nonlinear structures—an invariant manifold approach, *The Shock and Vibration Digest* 31 (1) (1994) 3–16.
[7] T.D. Burton, M.E. Young, Model reduction and nonlinear normal modes in structural dynamics, *ASME Winter Annual Meeting*, Chicago, IL, ASME, 1994.
[8] T.D. Burton, R.W. Rhee, The reduction of nonlinear structural dynamic models, *Journal of Vibration and Control* 6 (4) (2000) 531–556.
[9] K. Gallivan, E. Grimme, P. Van Dooren, Multi-point Padé approximants of large scale system via two-sided rational Krylov algorithm, in: *IEEE Conference on Decision and Control*, Lake Buena Vista, FL, 1994.
[10] P.M. Guilhen, P. Bertheier, G. Ferraris, M. Lalanne, Dynamic behavior of dissymmetric rotor bearings modelled periodic coefficient large system, in: *Proceedings of the 58th Shock and Vibration Symposium*, Huntsville, AL, 1987.
[11] S.C. Sinha, R. Pandiyan, Analysis of quasilinear dynamical systems with periodic coefficients via Liapunov–-Floquet transformation, *International Journal of Non-linear Mechanics* 29 (1994) 687–702.
[12] S.C. Sinha, R. Pandiyan, J.S. Bibb, Liapunov–Floquet transformation: computation and applications to periodic systems, *ASME Journal of Vibration and Acoustics* 118 (1996) 209–219.
[13] V. Deshmukh, S.C. Sinha, P. Joseph, Order reduction and control of parametrically excited dynamical systems, *Journal of Vibration and Control* 6 (2000) 1017–1028.

[14] S.C. Sinha, S. Redkar, E. Butcher, V. Deshmukh, Order reduction of nonlinear time periodic systems using invariant manifolds, in: *The Proceedings of 19th Biennial Conference on Mechanical Vibration and Noise, ASME 2003 DETC*, Chicago, IL, USA, September 2–6, 2003, 48445.
[15] V.A. Yakubovich, V.M. Starzhinski, *Linear Differential Equations with Periodic Coefficients*, vols. 1 and 2, Wiley, New York, 1975.
[16] R.M. Evan-Iwanowski, *Resonance Oscillations in Mechanical Systems*, Elsevier Scientific Publishing Company, Netherlands, 1976.
[17] C.S. Hsu, On the parametric excitation of a dynamic system having multiple degree of freedom, *Journal of Applied Mechanics* 30 (1963) 367–372.
[18] I.G. Malkin, Some Problems in the Theory of Nonlinear Oscillations, State Publishing House of Technical and Theoretical Literature, Moscow, 1956 (Translated from Russian).
[19] S.C. Sinha, S. Redkar, E. Butcher, Order reduction of nonlinear time periodic systems, in: *The Proceedings of Tenth International Congress on Sound and Vibration*, Stockholm, Sweden, July 7–10, 2003, pp. 2041–2048.